Modeliranje turbulencije u cilju primene numeričkih simulacija u hidrotehnici

Univerzitet u Beogradu

Građevinski fakultet

- Kurs Mehanike fluida na doktorskim studijama -

Nenad Jaćimović

Maj, 2013.

CFD – Computational Fluid Mechanics
Računska mehanika fluida

Primena metoda numeričke analize za rešavanje jednačina održanja:

- **Jednačina kontinuiteta**
- **Dinamička jednačina**
- **Jednačine transporta**

Zašto je potrebno „modeliranje“ turbulencije ?

Primer: Modeliranje dvofaznog strujanja voda / vazduh:

- **Mass conservation equations:**

$$\frac{\partial \alpha_w}{\partial t} + \frac{\partial(\alpha_w V_{wi})}{\partial x_i} = \frac{G_{gw}}{\rho_w}$$

$$\frac{\partial(\rho_g \alpha_g)}{\partial t} + \frac{\partial(\rho_g \alpha_g V_{gj})}{\partial x_j} = -G_{gw}$$

$$\alpha_w + \alpha_g = 1 \qquad \rho_g = \frac{p_g M_g}{RT}$$

α_W - water phase content

α_g – oxygen phase content

G_{gw} - Mass transfer of oxygen from the gas to the water phase

- **Momentum equations:**
  - water phase:

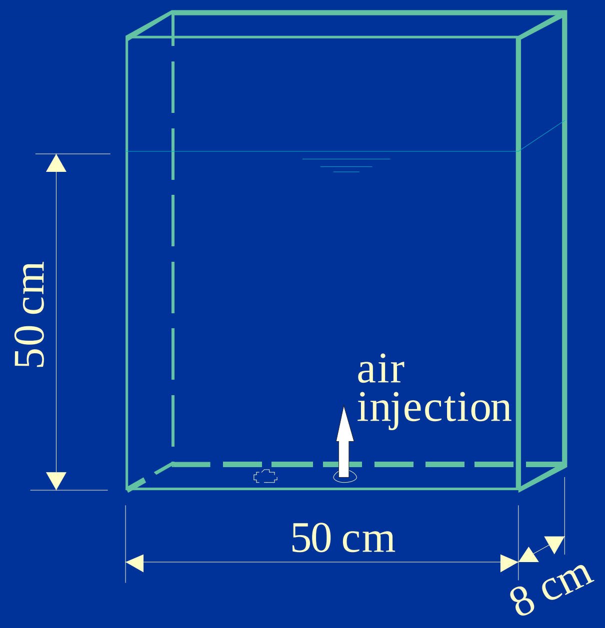
$$\alpha_w \frac{\partial V_{wi}}{\partial t} + \alpha_w V_{wj} \frac{\partial V_{wi}}{\partial x_j} = -\frac{\alpha_w}{\rho_w}\frac{\partial p_w}{\partial x_i} + \alpha_w g_i + \frac{1}{\rho_w}\frac{\partial(\alpha_w \tau_{ij})}{\partial x_j} + \frac{1}{\rho_w} F_{gw}$$

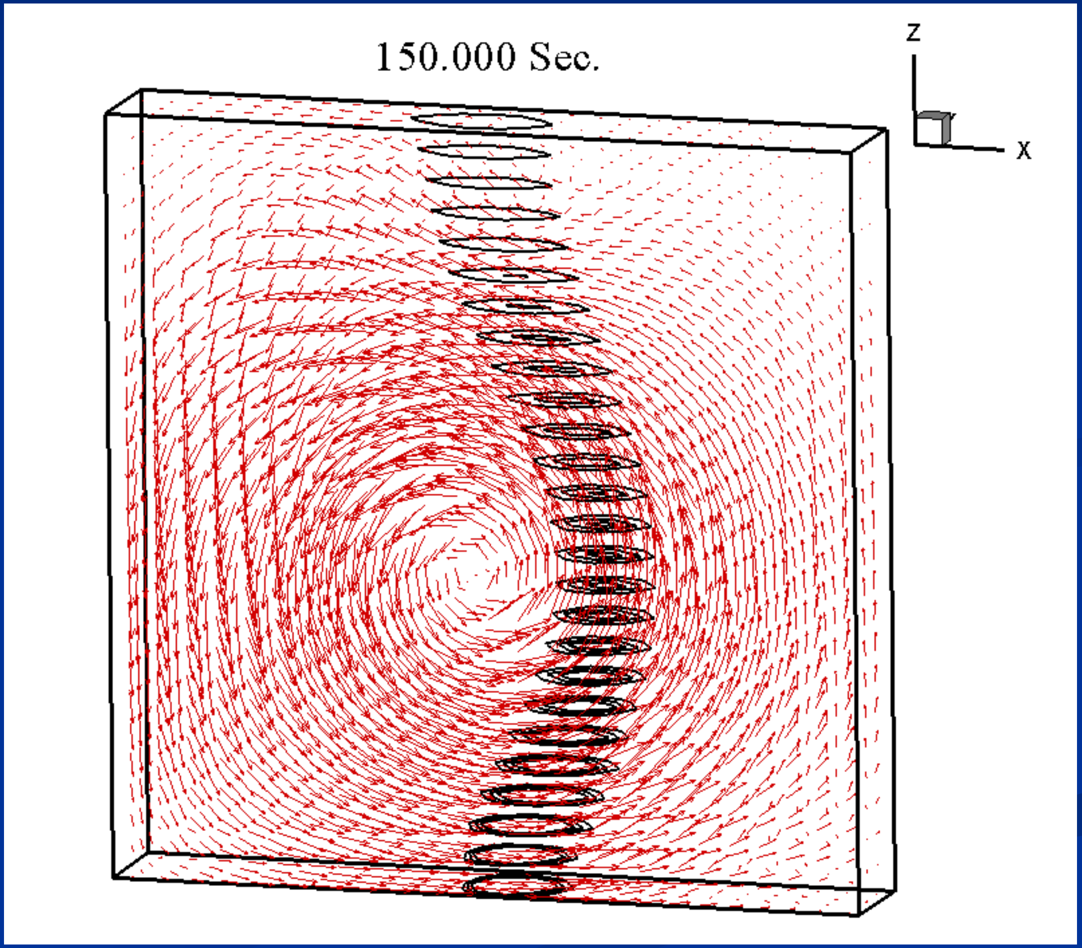
  - gas phase:

$$\frac{\partial(\alpha_g \rho_g V_{gi})}{\partial t} + \frac{\partial(\alpha_g \rho_g V_{gi} V_{gj})}{\partial x_j} = -\alpha_g \frac{\partial p_g}{\partial x_i} + \alpha_g \rho_g g_i - F_{gw}$$

Momentum exchange

Modeliranje dvofaznog strujanja voda/vazduh:

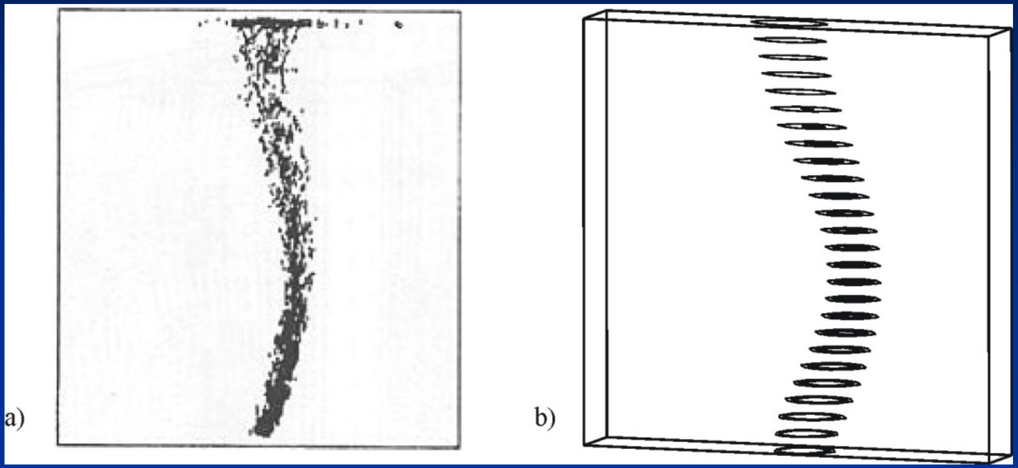




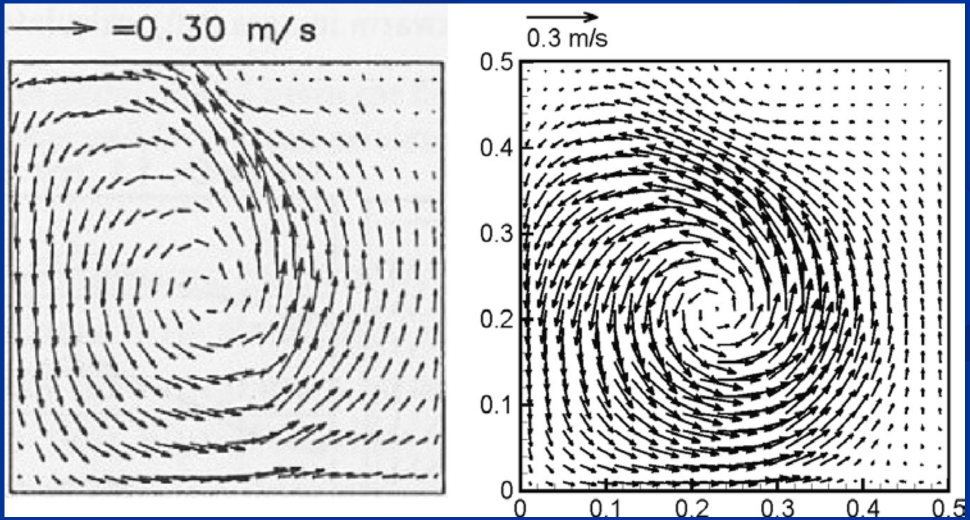
Q_{air}=1.0 L/min

Poređenje sa merenjima:

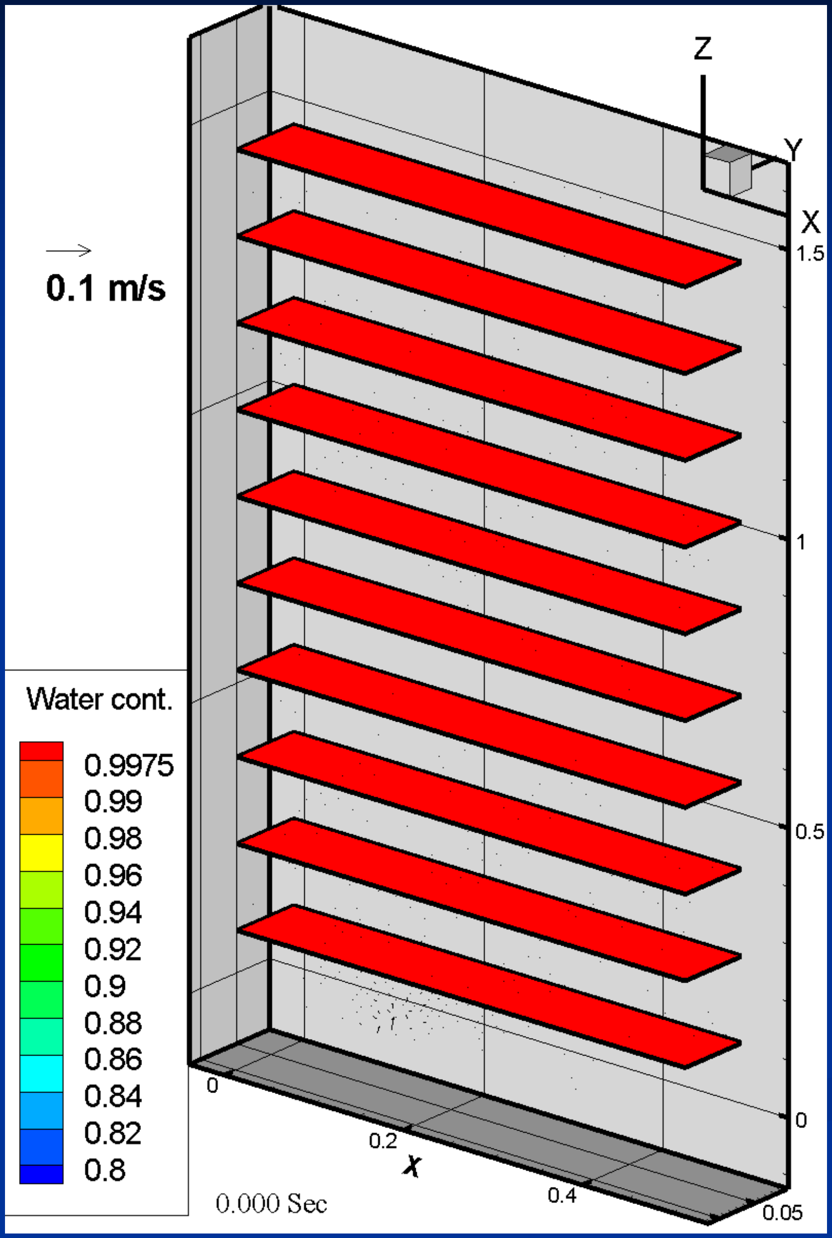
Poređenje oblika vazdušne struje



Poređenje strujne slike tečne faze

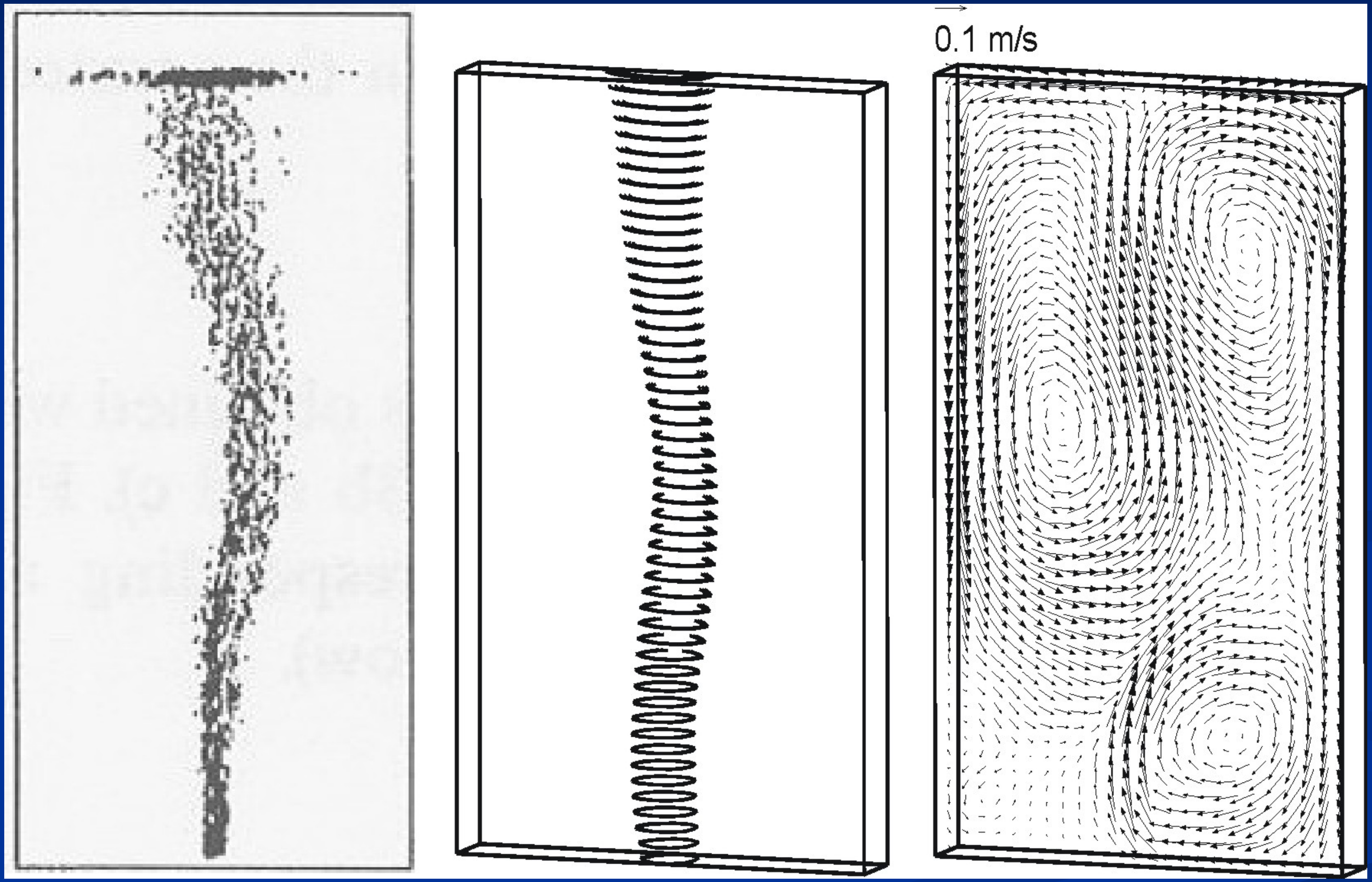


3D simulacija eksperimenta:



Osmotreni raspored mehurića vazduha:

Poređenje rezultata simulacije sa merenjima



$Q_{air} = 1\ L/min$

Rejnoldsove jednačine:

$$\rho\frac{\partial U_i}{\partial t}+\rho\frac{\partial\left(U_iU_j\right)}{\partial x_j}=-\frac{\partial p_w}{\partial x_i}+\rho g_i+\mu\frac{\partial^2 U_i}{\partial x_j\partial x_j}-\frac{\partial}{\partial x_j}\rho\overline{u_i^{'}u_j^{'}}$$

$$\frac{\partial U_i}{\partial x_i}=0 \qquad u_i=U_i+u_i^{'}$$

- **Modeliranje Rejnoldsovih napona predstavlja ključni element zatvaranja sistema jednačina, koje se potom rešavaju metodama numeričke analize.**

- **Dva su osnovna efekta turbulencije na glavni tok:**
  - **Oduzima energiju glavnog toka;**
  - **Doprinosi transportu mase, količine kretanja ili energije upravno na glavni tok.**

Dakle, efekti su isti kao i u slučaju molekularne viskoznosti (npr. kod laminarnog strujanja).

- Na osnovu toga, prirodno je pretpostaviti da se efekti turubulencije na glavni tok, predstavljeni Rejnoldsovim naponima u osrednjenim jednačinama, mogu „modelirati“ analogno viskoznim naponima (koncept turbulentne viskoznosti – Boussinesq/1877.).

$$-\overline{u_i^{'}u_j^{'}} = \nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k \qquad k = \frac{1}{2}\overline{u_i^{'}u_i^{'}}$$

- Na osnovu dimenzionalne analize može se zaključiti:

$$\nu_T = C\upsilon_T L_T$$

- Na ovaj način se problem modeliranja turbulencije sveo na problem ocene karakteristične brzine i dužine turbulencije u svakoj tački toka.

Na sličan način se može modelirati i transport rastvorene materije usled turbulentnih fluktuacija

- **Osrednjena jednačina „transporta“**

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \left(\Phi U_j \right)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(D \frac{\partial \Phi}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left(- \overline{u_j' \phi'} \right)$$

$$\phi = \Phi + \phi'$$

- **Turbulentna difuzija se može modelirati analogno molekularnoj difuziji:**

$$- \overline{u_j' \phi'} = D_T \frac{\partial \Phi}{\partial x_j}$$

gde se D_T obično daje u funkciji turbulentne viskoznosti:

$$D_T = \frac{\nu_T}{\sigma}$$

Malo istorije

- Prandtl (1925) – koncept dužine mešanja (algebarski izraz za Rejnoldsov napon);

- Prandtl (1945) – modeliranje turbulencije rešavanjem jedne dodatne dif. jednačine koja opisuje transport i promenu kinetičke energije turbulencije u toku;

- Kolmogorov (1942) – modeliranje turbulencije rešavanjem dve dodatne dif. jednačine koje opisuju transport i promenu kinetičke energije turbulencije i disipacije energije u toku (k-ω model);

$$L_T \approx \frac{k^{1/2}}{\omega}$$

k-ε model

(Harlow & Nakayama, 1968)
(Launder & Spalding, 1972)

- **Jednačina kinetičke energije fluktuacija (k jednačina)**

$$\rho\frac{\partial k}{\partial t}+\rho\frac{\partial\left(kU_j\right)}{\partial x_j}=-\rho\overline{u_i^{'}u_j^{'}}\frac{\partial U_i}{\partial x_j}-\mu\overline{\frac{\partial u_i^{'}}{\partial x_k}\frac{\partial u_i^{'}}{\partial x_k}}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}-\frac{1}{2}\rho\overline{u_i^{'}u_i^{'}u_j^{'}}-\overline{p^{'}u_j^{'}}\right)$$

lokalna i konv. promena k

produkcija k

disipacija k (ρε)

difuzija k

transport k usled fluktuacija brzine i pritiska

$$-\rho\overline{u_i^{'}u_j^{'}}=\mu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}\rho\delta_{ij}k$$

$$\mu_T=C_\mu\rho k^{1/2}L_T \qquad \varepsilon=C_D\frac{k^{3/2}}{L_T} \qquad \frac{\mu_T}{\sigma_k}\frac{\partial k}{\partial x_j}=\left(-\frac{1}{2}\rho\overline{u_i^{'}u_i^{'}u_j^{'}}-\overline{p^{'}u_j^{'}}\right)$$

ε jednačina

produkcija ε ($P\varepsilon$)

$$\rho\frac{\partial \varepsilon}{\partial t}+\rho\frac{\partial(\varepsilon U_j)}{\partial x_j}=-2\mu\left[\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}+\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_i}}\right]\frac{\partial U_i}{\partial x_j}-2\mu\overline{u_k'\frac{\partial u_i'}{\partial x_j}}\frac{\partial^2 U_i}{\partial x_k \partial x_j}-$$

lokalna i konv. promena ε

$$-2\mu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}-2\mu\nu\overline{\frac{\partial^2 u_i'}{\partial x_j \partial x_k}\frac{\partial^2 u_i'}{\partial x_j \partial x_k}}+$$

disipacija ε

$$+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \varepsilon}{\partial x_j}-\mu\overline{u_j'\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}-2\nu\overline{\frac{\partial p'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right)$$

difuzija ε

transport ε usled fluktuacija brzine i pritiska

Uvodi se pretpostavka o lokalnoj ravnoteži :

$$P_k \approx \varepsilon \qquad P_\varepsilon \approx \frac{P_k}{T_T} \qquad T_T \approx \frac{k}{\varepsilon}$$

produkcija ε ($P\varepsilon$)

$$P_\varepsilon = -2\mu\left[\overline{\frac{\partial u_i^{'}}{\partial x_k}\frac{\partial u_i^{'}}{\partial x_k}} + \overline{\frac{\partial u_k^{'}}{\partial x_i}\frac{\partial u_k^{'}}{\partial x_i}}\right]\frac{\partial U_i}{\partial x_j} - 2\mu\overline{u_k^{'}\frac{\partial u_i^{'}}{\partial x_j}}\frac{\partial^2 U_i}{\partial x_k \partial x_j} = C_{\varepsilon 1}\frac{k}{\varepsilon}P_k$$

disipacija ε

$$D_\varepsilon = -2\mu\overline{\frac{\partial u_i^{'}}{\partial x_j}\frac{\partial u_i^{'}}{\partial x_k}\frac{\partial u_j^{'}}{\partial x_k}} - 2\mu\nu\overline{\frac{\partial^2 u_i^{'}}{\partial x_j \partial x_k}\frac{\partial^2 u_i^{'}}{\partial x_j \partial x_k}} \approx \frac{\varepsilon}{T_T} \qquad D_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

disperzija ε

$$\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \varepsilon}{\partial x_j} - \mu\overline{u_j^{'}\frac{\partial u_i^{'}}{\partial x_k}\frac{\partial u_i^{'}}{\partial x_k}} - 2\nu\overline{\frac{\partial p^{'}}{\partial x_k}\frac{\partial u_j^{'}}{\partial x_k}}\right) = \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \varepsilon}{\partial x_j} + \frac{\mu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)$$

Rezime (k-ε model)

- ***k* jednačina:**

$$\rho\frac{\partial k}{\partial t}+\rho\frac{\partial\left(kU_j\right)}{\partial x_j}=P_k-\rho\varepsilon+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}+\frac{\mu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right)$$

- ***ε* jednačina:**

$$\rho\frac{\partial \varepsilon}{\partial t}+\rho\frac{\partial\left(\varepsilon U_j\right)}{\partial x_j}=C_{\varepsilon 1}\frac{\varepsilon}{k}P_k-C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \varepsilon}{\partial x_j}+\frac{\mu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)$$

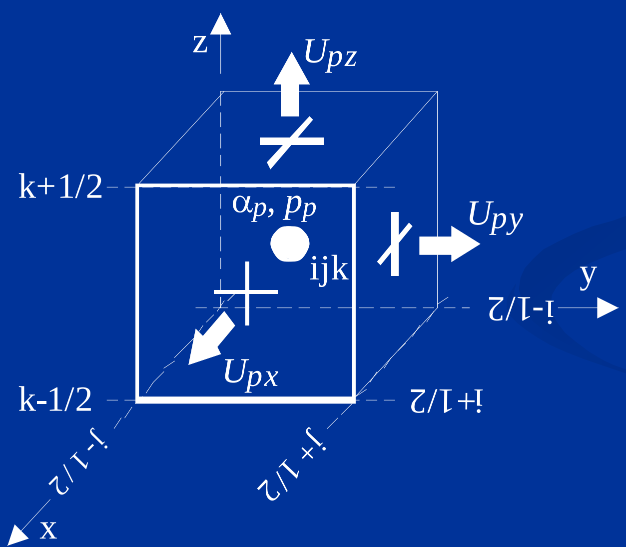
$$\mu_T=C_\mu\rho\frac{k^2}{\varepsilon}$$

$$C_{\varepsilon 1}=1.44 \qquad C_{\varepsilon 2}=1.92 \qquad C_\mu=0.09 \qquad \sigma_k=1.0 \qquad \sigma_\varepsilon=1.3$$

Sada, kada znamo kako „modelirati“ turbulenciju, preostaje numeričko rešavanje osnovnih jednačina

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}^{\mu} + \tau_{ij}^{T}\right)$$

$$\frac{\partial U_i}{\partial x_i} = 0$$



- ➢ **HSMAC shema za proračun nepoznatih brzina i pritisaka u strujnom polju**

Iterativna shema se sastoji iz tri koraka:

- ***1. korak*:**

$$\frac{(U_i)^* - (U_i)^n}{\Delta t} = (inertia\ t.)^n + (pressure\ t.)^{n+1} + (viscous\ t.)^n + (turb\ t.)^n$$

- ***2. korak*:**

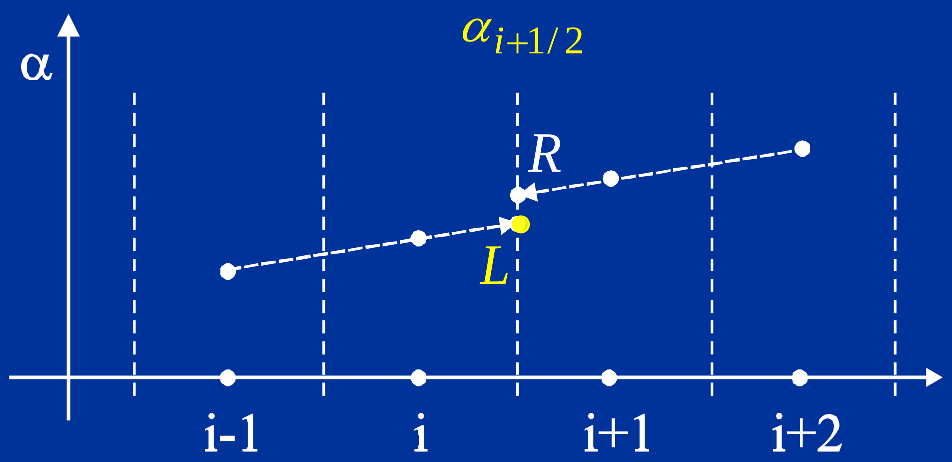
$$D = \frac{\partial U_i^*}{\partial x_i}$$

- ***3. korak*:**

$$p^{n+1,r+1} = p^{n+1,r} - \omega D \Big/ \left(\frac{\partial D}{\partial p}\right) = p^{n+1,r} - \frac{\omega D^{r+1}}{2\Delta t\left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2} + \frac{1}{\Delta x_3^2}\right)}$$

Problem konvektivnih članova u jednačinama (ili kako umanjiti numeričku difuziju, a pri tome izbeći oscilacije rešenja)

- **Sheme višeg reda tačnosti sa tzv. TVD limiterima:**

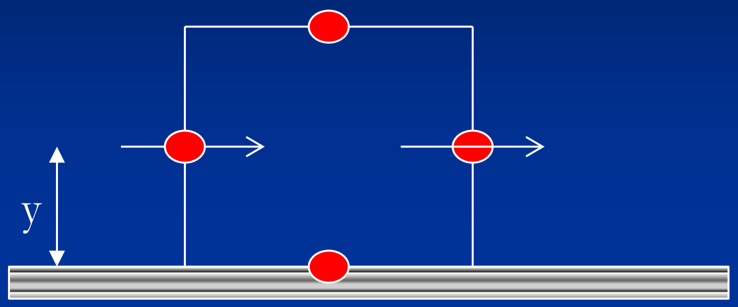


$$\alpha^{L}_{i+1/2} = a_i + \frac{1}{2} S$$

"Superbee" limiter:

$$S = \max \begin{cases} 0 \\ \min\left[2(\alpha_{i+1} - \alpha_i), (\alpha_i - \alpha_{i-1})\right] \\ \min\left[(\alpha_{i+1} - \alpha_i), 2(\alpha_i - \alpha_{i-1})\right] \end{cases}$$

Granični uslov uz čvrstu granicu



- **gladak zid:**

$$\frac{U(y)}{U_*} = \frac{1}{\kappa}\ln\left(\frac{yU_*}{\nu}\right) + 5.5$$

- **hrapav zid:**

$$\frac{U(y)}{U_*} = \frac{1}{\kappa}\ln\left(\frac{y}{k_s}\right) + 8.5$$

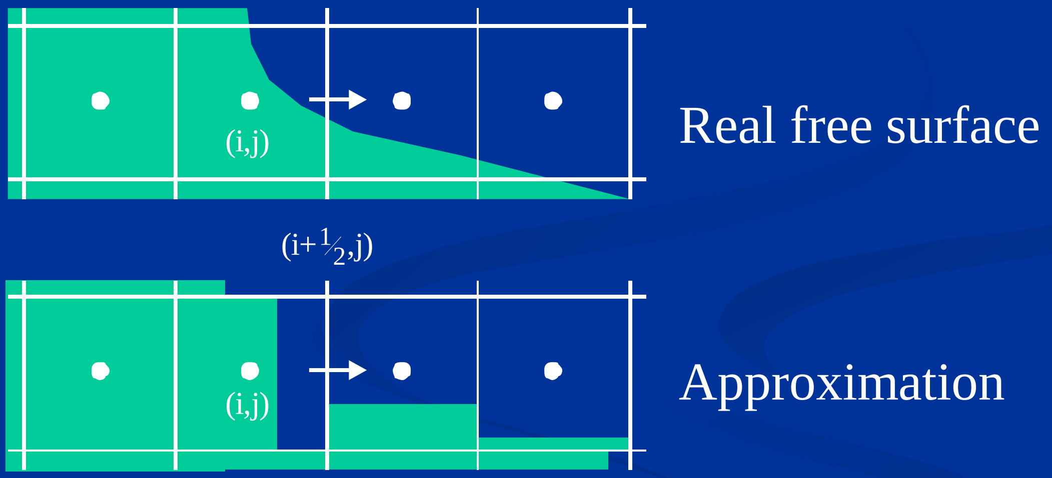
$$k = \frac{U_*^2}{\sqrt{C_\mu}} \qquad \varepsilon = \frac{U_*^3}{y\kappa}$$

Problem praćenja slobodnog nivoa vode:

- **MAC (Welch et al., 1965.)**
- **Density function method (Asai & Tsubogo, 2005)**
- **VOF metod (Hirt & Nichols, 1981.):**

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_i}\left[F U_i\right] = 0$$

F – cell saturation



$$q_{i+1/2,j} = MIN\left\{F_{i+1,j} U_{i+1/2,j}\delta t + MAX\left[0.0, (1.0 - F_{i+1,j})U_{1+1/2,j}\delta t - (1.0 - F_{i,j})\delta x\right], F_{i,j}\delta x\right\}$$

Ukoliko se "nema vremena" za programiranje, postoje i besplatna rešenja

- **iRIC software :** http://i-ric.org/en/